RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIFTH SEMESTER EXAMINATION, DECEMBER 2018

THIRD YEAR [BATCH 2016-19]

MATHEMATICS [Honours]

Paper : V

Date : 17/12/2018
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each Group]

Group – A

Answer any five questions from Question Nos. 1 to 8 : [5×10]

1. a) Prove that A_4 has no subgroup of order 6.
   b) Let H be a normal subgroup of G and o(H) = 2. Prove that $H \subseteq Z(G)$.
   c) Suppose G is a finite cyclic group of order n. Prove that Aut G is a group of order $\varnothing(n)$, where $\varnothing$ stands for Euler's Phi function. [3+2+5]

2. a) State Cayley's theorem.
   b) Show that there are only two noncommutative groups of order 8 upto isomorphism.
   c) Show that the additive group ($\mathbb{Z}$, +) cannot be expressed as an internal direct product of two nontrivial subgroups. [2+5+3]

3 a) Establish class equation for a finite group. Write class equation for S_3.
   b) Find the number of elements of order 3 in a noncyclic group of order 21.
   c) Write true or false with proper justification of the following statement :
   Any epimorphism of ($\mathbb{Z}$, +) onto ($\mathbb{Z}$, +) is an isomorphism. [5+2+3]

4. a) Suppose G is a group of order 96. Show that G has a normal subgroup of order 16 or 32.
   b) Suppose G is a group of order 90. Show that G is not simple. [5+5]

5. a) Give an example of a field F and subfields K_1, K_2 such that $K_1 \cup K_2$ is not a subfield of F.
   b) If X is an infinite set and $\mathcal{A} = \{A \in \mathcal{P}(X) : A \text{ is finite}\}$ then show that $(\mathcal{P}(X), \Delta, \cap)$ is a ring with identity and $\mathcal{A}$ is a subring of $\mathcal{P}(X)$ without identity, here $\mathcal{P}(X)$ denotes the power set of X.
   c) Show that $\mathbb{Z}\left[\sqrt{3}\right]$ and $\mathbb{Z}\left[\sqrt{5}\right]$ are not isomorphic as rings. [2+4+4]

6. a) (i) Find the characteristic of the ring $\mathbb{Z}_4[x]$, the polynomial ring over $\mathbb{Z}_4$.
   (ii) In $\mathbb{Z}_8[x]$, show that $\bar{4}x + \bar{3}$ is a unit.
   b) Show that every Euclidean domain is a P.I.D.
   c) If a and b are two elements in an Euclidean Domain R with valuation 'd' and b is invertible then prove that d (ab) = d (a) [(2+2)+4+2]

7. a) Prove that 3 is irreducible but not prime in the integral domain $\mathbb{Z}[i\sqrt{5}] = \{a + bi\sqrt{5} : a, b \in \mathbb{Z}\}$.
   b) Let R be an Integral Domain in which
   (i) Every $a \in R - \{0\}$ which is not a unit can be expressed as a product of irreducible elements.
   (ii) Every irreducible element is prime.
   Prove that R is a U.F.D. [5+5]

8. a) Prove that if R is a commutative ring with 1, then R contains a maximal ideal.

b) Show that the ideal <x> is prime in $\mathbb{Z}[x]$.

c) Prove that the ideal I $=\{(5x, y) : x, y \in \mathbb{Z}\}$ is a maximal ideal in $\mathbb{Z}\times\mathbb{Z}$. [5+2+3]

Group – B

Answer any six questions from Question Nos. 9 to 17 **:** [6×5]

9. If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, and if the one-dimensional limit $\lim_{y\to b} f(x,y)$ exists, prove that [5]

$$\lim_{x\to a}\left[\lim_{y\to b} f(x,y)\right] = L.$$

10. a) Consider the function $f(x,y) = \dfrac{\sin x + \sin 2y}{\tan 2x + \tan y}$. Show that for this function repeated limits exist at (0,0) but double limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

b) Give an example of a function for which the double limit exists at a point but repeated limits do not exist. [3+2]

11. Let f be a function from $\mathbb{R}^n$ into $\mathbb{R}^m$, and $f = (f_1, f_2, ...f_m)$. Prove that f is differentiable at a point c iff each f_i is differentiable at c. [5]

12. Let f: S$\to\mathbb{R}$, where S is an open set in $\mathbb{R}^2$, be a function. Let (a,b)$\in$ S & both $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ are differentiable at (a,b). Show that $f_{xy}(a,b)=f_{yx}(a,b)$. [5]

13. Let (x,y) approach (0,0) along $y=-x$. Find $\lim \dfrac{\sin xy + xe^x - y}{x\cos y + \sin 2y}$ using Taylor's theorem. [5]

14. Let f: $\mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. By considering the function
$g(t) := f[ty_1+(1-t)x_1, y_2] + f[x_1, ty_2+(1-t)x_2]$
prove that
$f(y_1, y_2) - f(x_1, x_2) = (y_1 - x_1)D_1 f(z_1, y_2) + (y_2 - x_2)D_2 f(x_1, z_2)$, where z_i lies in between x_i and y_i. [5]

15. State inverse function theorem and implicit function theorem in general. [2+3]

16. Verify that the function $f(x,y,z)=¼(x^4+y^4+z^4) - xyz$ has a stationary point at (1,1,1) and determine the nature of this stationary point by computing the eigenvalues of its Hessian matrix. [5]

17. Use Lagrange's method to find the shortest distance from the point (0,3) to the parabola $x^2 - 4y = 0$. [5]

Answer <u>any four</u> questions from <u>Question Nos. 18 to 23</u> : [4×5]

18. Show that a function of bounded variation is bounded but the converse is not true. [3+2]

19. Prove that a function satisfying Lipchitz condition on [a,b] is a function of bounded variation on [a,b] . Is the converse true? Explain. [3+2]

20. Let $f : [a,b] \to \mathbb{R}$ be bounded on [a,b] & let f be continuous on [a,b] except on a subset S such that the number of limit points of S is finite. Prove that f is integrable on [a,b] [5]

21. A function f is defined over the closed interval [1,3] as follows:

    f(x)= 1, $1 \le x < 2$

    =2, $2 \le x \le 3$

    State with reason:

    (i) Whether $\int_1^3 f(x)dx$ exists.

    (ii) Whether the known result of integral calculus $\int_a^b f(x)dx = (b-a)f(c)$ for $c \in [a,b]$ holds in this example.

    (iii) Whether the fundamental theorem of integral calculus is applicable to f(x) in [1, 3]. [1+2+2]

22. (a) Defining e as $\int_1^e \frac{dt}{t} = 1$ show that $2 < e < 3$.

    (b) Using Bonnet's form of 2nd Mean Value theorem of integral calculus, show that if b>a>0 then $\left|\int_a^b \frac{\sin x}{x} dx\right| \le 2/a$. [3+2]

23. Show that $\frac{\pi^3}{24\sqrt{2}} < \int_0^{\pi/2} \frac{x^2}{\sin x + \cos x} dx < \frac{\pi^3}{24}$. [5]

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